Fundamental group for some cuspidal curves *

José Ignacio Cogolludo †

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Introduction

In [1], Hirano gives a method to construct families of curves with a large number of singularities. The idea is to consider an abelian covering of $\mathbf{P}^2$ ramified along three lines in general position and to take the pull-back of a curve C intersecting the lines non-generically. Similar constructions are used by Shimada in [10] and Oka in [8]. We apply this method for the case where C is a conic, constructing a family of curves with the following asymptotic behavior (see [9])

$$\lim_{n\to\infty}\Big(\sum_{p\in Sing\tilde{C}_n}\mu(p)\Big)\Big/\Big(\deg(\tilde{C}_n)\Big)^2=\frac{3}{4}.$$

The goal of this paper is to calculate the fundamental group for the curves in this family as well as their Alexander polynomial. We have the following

Theorem 1 *Let $f(x,y,z)=x^2+y^2+z^2+2(xz-xy+yz)$ and $\tilde{C}_n$ be the projective plane curve defined by $\{[x:y:z]\in\mathbf{P}^2 \mid f(x^n,y^n,z^n)=0\}$ then*

$$\pi_1(\mathbf{P}^2\setminus\tilde{C}_n) \;=\; <\varepsilon_{i,j}\;;\quad \begin{array}{rcl} \varepsilon_{i+2,j} &=& \varepsilon_{i+1,j}^{-1}\cdot\varepsilon_{i,j}\cdot\varepsilon_{i+1,j}, \\ \varepsilon_{i,j+2} &=& \varepsilon_{i,j+1}^{-1}\cdot\varepsilon_{i,j}\cdot\varepsilon_{i,j+1}, \qquad i,j\in\mathbf{Z}/n\mathbf{Z} \\ \multicolumn{3}{l}{\varepsilon_{0,0}\cdot\varepsilon_{0,1}\cdot\varepsilon_{1,1}\cdot\ldots\cdot\varepsilon_{n-1,n-1}\cdot\varepsilon_{n-1,0}=1} \end{array} \quad >.$$

All curves $\tilde{C}_{2k}$ are reducible. The first irreducible member of this family, that is $\tilde{C}_3$, is the sextic with 9 ordinary cusps considered by Zariski in [11]. He gave a presentation of its fundamental group (also see [2]). We show explicitly an isomorphism between ours and Zariski's presentation. The global Alexander polynomial of this sextic, $(t^2-t+1)^3$ (see [4]), is different from the local Alexander polynomial of its singularities, (t^2-t+1). This was the first and unique example in the literature. Our family provides infinitely many examples of this behavior for curves containing any type of locally irreducible A_{2k}-singularity. We have the following

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Theorem 2 *Let $\tilde{C}_n$ be a curve defined as above, then the global Alexander polynomial of $\tilde{C}_{2k+1}$ is*

$$\Delta_{\tilde{C}_{2k+1}}(t) = (t^{2k} - t^{2k-1} + ... + t^2 - t + 1)^3.$$

For n odd, One can also relate the groups $\pi_1(\mathbf{P}^2 \setminus \tilde{C}_n)$ with the free products $(\mathbf{Z}/2\mathbf{Z})*(\mathbf{Z}/n\mathbf{Z})$ considered by Oka in [7]. The latter correspond to fundamental groups of the curves defined by $\{[x:y:z] \in \mathbf{P}^2 \mid (y^n - z^n)^2 + (x^2 - y^2)^n = 0\}$. Our groups are non-central extensions of these. In fact, we can obtain the Oka curves as a perturbation of ours.

For the case $n = 3$, this perturbation provides sextics with six ordinary cusps on a conic. The special position of their singularities results in the following, these sextics and those with six cusps in general position belong to different connected components, as Zariski showed in [12]. Our construction provides a degeneration from the family of sextics with six ordinary cusps on a conic to the family of sextics with nine cusps. The latter family is connected, since any curve in it is dual to a non-singular cubic.

A more detailed summary of this article is the following: In the first section we apply the Hirano method for a conic C tangent to three lines, obtaining a family of curves with A_n-singularities where n depends on the degree of the covering. In the second section we calculate the fundamental group of the curves of this family and in the third section we find their Alexander polynomials. In section four we show how this family can be obtained also as a degeneration of the Oka curves described above and how the change of the fundamental groups can be detected by deforming the original curve C.

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1 Construction of the family of curves $\tilde{C}_n$

Let's consider three lines L_0, L_1 and L_2 in general position and a non-singular conic $C \equiv \{f = 0\}$ in $\mathbf{P}^2$ tangent to the lines. By changing coordinates on the plane we can assume L_i to be the line $\{[x_0 : x_1 : x_2] \mid x_i = 0\}$, we define a map on $\mathbf{P}^2$ as follows

$$\begin{array}{ccccc} [x_0 : x_1 : x_2] & \in & \mathbf{P}^2 & \\ \downarrow & & \downarrow & p_n \\ [x_0^n : x_1^n : x_2^n] & \in & \mathbf{P}^2 & \end{array}$$

for any natural number n.

This is an abelian regular covering outside the lines L_0, L_1 and L_2. Its Galois group is $(\mathbf{Z}/n\mathbf{Z}) \oplus (\mathbf{Z}/n\mathbf{Z})$. On the lines it has degree n except on the

double points, where the map is one-to-one. Let's denote by $\tilde{L}_i$ the pre-image of the line L_i. Now consider the pull-back of C, which is an algebraic curve defined by the equation $\tilde{C}_n \equiv \{f(x_0^n : x_1^n : x_2^n) = 0\}$. Its degree is $2n$ and it has the following properties:

(1) *It has $3n$ singular points of local type A_{n-1} and no other singularities.*

Proof. Since the covering is unramified outside the three lines $\tilde{L}_0, \tilde{L}_1$ and $\tilde{L}_2$, the curve $\tilde{C}_n$ is non-singular outside this locus. Therefore it can only be singular on the intersection with the pre-image of the branching locus. This corresponds to the pre-image of the three points of intersection of C and $L_0 \cup L_1 \cup L_2$, say p_0, p_1 and p_2. Indeed this pre-image consists of $3n$ points. The equation of C around any of the point p_i can be written locally as $y^2 = x$, where $\{x = 0\}$ is the local equation of the branching line L_i. Since the ramification order is n, the local equation for the pre-images is $y^2 = x^n$, that is, singular points of type A_{n-1}. $\square$

(2) *If n is odd $\tilde{C}_n$ is irreducible.*

Proof Since all singularities of $\tilde{C}_n$ are locally irreducible, the curve must be irreducible. $\square$

(3) *If n is even $\tilde{C}_n$ has exactly four non-singular irreducible components.*

Proof If $n = 2$ it turns out that $\tilde{C}_n$ has degree 4 and 12 singular points of type A_1, that is, ordinary double points. By the genus formula $\tilde{C}_n$ cannot be irreducible, and by Bezout theorem, $\tilde{C}_n$ cannot be the union of two non-singular conics. Similar considerations eliminate the cases of $\tilde{C}_n$ being a line and an irreducible cubic or a conic and two lines. Therefore, it must be an arrangement of lines intersecting transversely.

If $n = 2k$, the covering factors through $p_n = \tilde{p}_k \circ p_2$. Therefore $\tilde{C}_n$ will be the pre-image by the abelian covering $\tilde{p}_k$ of four lines intersecting, transversely, the ramification locus. Since transverse intersections with the ramification locus do not generate singularities on their pre-images, the pull-back of each line is a non-singular curve and therefore irreducible. $\square$

2 Fundamental group of $\mathbf{P}^2 \setminus \tilde{C}_n$

The first step will be to calculate a presentation for

$$G := \pi_1(\mathbf{P}^2 \setminus (C \cup L_0 \cup L_1 \cup L_2)).$$

Since we have a real picture of this arrangement where all singular points appear, we can proceed rather easily by using the Zariski-Van Kampen method.

To start with, we consider the line L of Fig. 1.

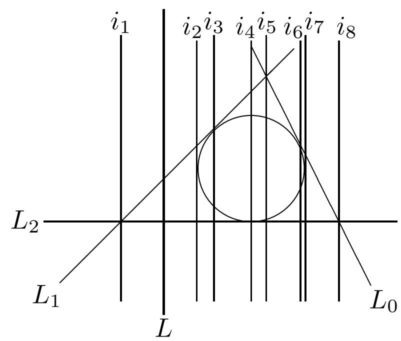

Fig. 1

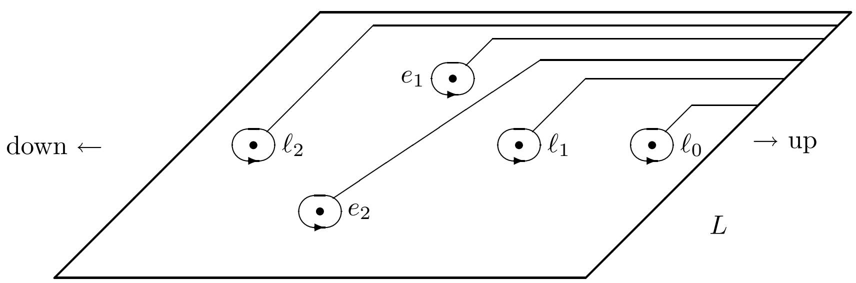

Fig. 2

The paths e_1, e_2, ℓ_0, ℓ_1 and ℓ_2 of Fig. 2 correspond to a system of generators for G. A system of relators is given by the action of the monodromy around the exceptional lines $i_1, ..., i_8$. There is a canonical way of considering this action which comes from the fact that the arrangement is real [6]. Proceeding in this way, the relations look as follows:

$$e_1 = e_2 = e \quad \text{(around } i_2\text{);}$$
$$[\mathrm{e}\ell_1\mathrm{e}^{-1}, \ell_2] = 1 \quad \text{(around } \mathrm{i}_1\text{);}$$
$$(\ell_1 e)^2 = (e\ell_1)^2 \quad \text{(around } i_3\text{);}$$
$$(\ell_2 e)^2 = (e\ell_2)^2 \quad \text{(around } i_4\text{);}$$
$$[\mathrm{e}\ell_1\mathrm{e}^{-1}, \ell_0] = 1 \quad \text{(around } \mathrm{i}_5\text{);}$$
$$(\ell_0 \cdot \ell_1^{-1} e\ell_1)^2 = (\ell_1^{-1} e\ell_1 \cdot \ell_0)^2 \quad \text{(around } i_6\text{);}$$
$$\ell_0^{-1}\ell_1^{-1} e\ell_1\ell_0 = \ell_2 e\ell_2^{-1} \quad \text{(around } i_7\text{);}$$
$$[\ell_2, \ell_1^{-1}\mathrm{e}\ell_1\ell_0\ell_1^{-1}\mathrm{e}^{-1}\ell_1] = 1 \quad \text{(around } \mathrm{i}_8\text{);}$$
$$\ell_2 \mathrm{e}^2\ell_1\ell_0 = 1 \quad \text{(from the projective situation.)}$$

This presentation can be simplified to get the following

$$G \;=\; <\; e, \ell_1, \ell_2 \;;\quad \begin{array}{l} [e\ell_1 e^{-1}, \ell_2] = 1, \\ (\ell_1 e)^2 = (e\ell_1)^2,\; (\ell_2 e)^2 = (e\ell_2)^2, \\ (\ell_1\ell_2 e)^2 = (e\ell_1\ell_2)^2 \end{array} \quad > .$$

Since $p_{n|\mathbf{P}^2\setminus(\tilde{L}_0\cup\tilde{L}_1\cup\tilde{L}_2)}$ is a topological covering, we have the following short exact sequence

$$1 \to K \to G \to S = \mathbf{Z}/n\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z} \to 1,$$

where S is isomorphic to the commutative subgroup of S_{n^2} generated by the images of ℓ_1 and ℓ_2, and K is the fundamental group of $\mathbf{P}^2 \setminus (\tilde{C} \cup \tilde{L}_0 \cup \tilde{L}_1 \cup \tilde{L}_2)$.

Our group $\pi_1(\mathbf{P}^2 \setminus \tilde{C}_n)$ is the quotient of K by the normal subgroup generated by the meridians of the three lines, that is, ℓ_1^n, ℓ_2^n and $\ell_0^n = (\ell_2 e^2 \ell_1)^{-n}$. In order to make things easier, and since we are not interested in K, but in a quotient, we will use these new relations in the coming calculations.

To use Reidemeister-Schreier method [5], we need a set of representatives for cosets of G with respect to K, say $\{\ell_1^i \ell_2^j\}_{i,j=0,\ldots,n-1}$, which gives rise to a system of generators for K

$$\begin{array}{lcl} e_{i,j} & := & (\ell_1^i\ell_2^j)\cdot e \cdot (\ell_1^i\ell_2^j)^{-1} \\ \ell_{1,i,j} & := & (\ell_1^i\ell_2^j)\cdot \ell_1 \cdot (\ell_1^{i+1}\ell_2^j)^{-1} \\ \ell_{2,i} & := & \ell_1^i \cdot \ell_2^n \cdot \ell_1^{-i}, \end{array}$$

where $i, j \in \mathbf{Z}/n\mathbf{Z}$.

Observe that the meridians of the lines $\tilde{L}_1$ and $\tilde{L}_2$ become $\ell_{1,n-1,0}$ and $\ell_{2,i}$. Therefore we can eliminate these generators whenever they appear in calculations.

To simplify the notation, let's denote

$$\begin{array}{l} e_{i,j}^{r,s} := e_{r,s}^{-1}\cdot e_{i,j}\cdot e_{r,s}, \qquad \text{and} \\ \varepsilon_{i,j} := e_{i,j}^{i,0}. \end{array}$$

In order to obtain a system of relators for K we use the rewriting method for the relators of G. So, for $[e\ell_1 e^{-1}, \ell_2] = 1$ we get

$$e_{i,j}\cdot \ell_{1,i,j}\cdot e_{i+1,j}^{-1}\cdot e_{i+1,j+1}\cdot \ell_{1,i,j+1}\cdot e_{i,j+1}^{-1} = 1,$$

which allows us to eliminate all generators of the form $\ell_{1,i,j}$ by

$$\ell_{1,i,j} = e_{i,j}^{-1}\cdot e_{i,0}\cdot e_{i+1,0}^{-1}\cdot e_{i+1,j}. \tag{1}$$

Using (1) we can rewrite $(\ell_1 e)^2 = (e\ell_1)^2$ as

$$\varepsilon_{i+2,j} \;=\; \varepsilon_{i,j}^{i+1,j},$$

and $(\ell_2 e)^2 \;=\; (e\ell_2)^2$ as

$$e_{i,j+2} \;=\; e_{i,j}^{i,j+1},$$

or equivalently

$$\varepsilon_{i,j+2} \;=\; \varepsilon_{i,j}^{i,j+1}.$$

The relations coming from $(\ell_1\ell_2 e)^2 = (e\ell_1\ell_2)^2$ are all consequences of the previous ones.

It only remains to add the last relation for the meridian ℓ_0^n, which can be written as

$$\varepsilon_{0,0}\cdot\varepsilon_{0,1}\cdot\varepsilon_{1,1}\cdot\ldots\cdot\varepsilon_{n-1,n-1}\cdot\varepsilon_{n-1,0} = 1.$$

Therefore, we have obtained the desired presentation

$$\pi_1(\mathbf{P}^2\setminus\tilde{C}_n) \;=\; <\varepsilon_{i,j}\;;\quad \begin{array}{l} \varepsilon_{i+2,j} \;=\; \varepsilon_{i,j}^{i+1,j}, \\ \varepsilon_{i,j+2} \;=\; \varepsilon_{i,j}^{i,j+1}, \qquad i,j\in\mathbf{Z}/n\mathbf{Z} \\ \varepsilon_{0,0}\cdot\varepsilon_{0,1}\cdot\varepsilon_{1,1}\cdot\ldots\cdot\varepsilon_{n-1,n-1}\cdot\varepsilon_{n-1,0}=1 \end{array} >. \qquad (2)$$

Note that it is fairly clear, by (2), how the parity of n determines that the abelianization of $\pi_1(\mathbf{P}^2\setminus\tilde{C}_n)$ is $\mathbf{Z}/2n\mathbf{Z}$ or $\mathbf{Z}^3\oplus\mathbf{Z}/\frac{n}{2}\mathbf{Z}$.

Observe that one can rewrite the presentation so that the elements ε_{ij} with $i,j=0,1$ generate the group.

The case $n=3$ provides another presentation of the fundamental group of a sextic with 9 cusps, different to the one given by Zariski [11] or Kaneko [2]. In fact they calculate the fundamental group as the kernel of the projection of the braid group with three strings on the torus, $B_3(T)$, onto the first homology group $H_1(T,\mathbf{Z})$. The presentation had generators

$$g_2,\; g_{00},\; g_{01},\; g_{10}\;\text{ and }\; g_{11}$$

and relations

$$g_2\cdot g_{ij}\cdot g_2 = g_{ij}\cdot g_2\cdot g_{ij}\;\text{ for }\; i,j=0,1,2,$$

where

$$\begin{array}{l} g_{20}=g_{10}\cdot g_{00}\cdot g_{10}^{-1} \\ g_{21}=g_{11}\cdot g_{10}\cdot g_{11}^{-1} \\ g_{02}=g_{01}\cdot g_{00}\cdot g_{01}^{-1} \\ g_{12}=g_{11}\cdot g_{10}\cdot g_{11}^{-1} \\ g_{22}=g_{21}\cdot g_{20}\cdot g_{21}^{-1}\;. \end{array}$$

An isomorphism between both presentations can be given by

$$\begin{array}{rcl} \varepsilon_{00} & \mapsto & g_2\cdot g_{11}\cdot g_2^{-1} \\ \varepsilon_{10} & \mapsto & g_2 \\ \varepsilon_{01} & \mapsto & g_{10} \\ \varepsilon_{11} & \mapsto & g_2\cdot g_{01}\cdot g_2^{-1} \\ \varepsilon_{11}\cdot\varepsilon_{00}^{01}\cdot\varepsilon_{11}^{-1} & \mapsto & g_{00}\;. \end{array}$$

3 Alexander polynomial of $\tilde{C}_n$ for n odd.

By Theorem [5.1.] in [3], we only have to calculate the superabundance

$$s_\kappa(2n-3-2n\kappa) = \dim H^1(\mathcal{A}_\kappa(2n-3-2n\kappa)),$$

where κ is a constant of quasi-adjunction corresponding to a singularity of type A_{n-1}, say $\kappa = \frac{n-2}{2n}$. Observe that, for this constant, the local ideal of quasi-adjunction $(\mathcal{A}_\kappa)_{m_p}$ imposes the weakest conditions on the linear system, in fact, it is just the maximal ideal of the local ring at the singular points p of $\tilde{C}_n$.

On one hand, the expected dimension of the linear system of curves of degree $n-1$ passing through $3n$ points in general position is

$$\frac{n(n+1)}{2} - 3n.$$

On the other hand, since all singularities belong to a cubic, the dimension of the linear system $\Gamma(\mathbf{P}^2, \mathcal{A}_\kappa(n-1))$ is

$$\frac{(n-3)(n-2)}{2}.$$

Therefore,

$$s_\kappa(2n-3-2n\kappa) = s_{\frac{n-2}{2n}}(n-1) = \frac{(n-3)(n-2)}{2} - \Big(\frac{n(n+1)}{2} - 3n\Big) = 3.$$

Hence the Alexander polynomial of $\tilde{C}_n$ is

$$\Delta_{\tilde{C}_n}(t) = (t^{n-1} - t^{n-2} + ... + t^2 - t + 1)^3. \tag{3}$$

One can also use (2) to give a presentation of the commutator and calculate the exponent of (3) as the rank of its abelianization.

4 A deformation of the family

Instead of considering the conic and the three tangent lines, we will move one of the lines so that it becomes transverse. There is a surjective map

$$\pi_1(\mathbf{P}^2 \setminus (C \cup L_1 \cup L_2 \cup L_3)) \longrightarrow \pi_1(\mathbf{P}^2 \setminus (C \cup L_1 \cup L_2 \cup L_3')). \tag{4}$$

The same Hirano construction as before will give us a family of curves $\tilde{C}_n'$ of degree $2n$ but with $2n$ singularities of type A_{n-1} instead of $3n$. We will have a surjection

$$\pi_1(\mathbf{P}^2 \setminus \tilde{C}_n) \longrightarrow \pi_1(\mathbf{P}^2 \setminus \tilde{C}_n'). \tag{5}$$

One can write down the map (4) and apply the Reidemeister-Schreier method simultaneously to get the surjective map (5). One sees that it comes from adding the relations

$$\varepsilon_{i,j} = \varepsilon_i \text{ for any } j = 0, ..., n-1.$$

That is, the non-abelian group $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/n\mathbf{Z})$ given in [7]. Actually the curves $C_{2,n}$ presented by Oka are a deformation of the $\tilde{C}'_n$. If we consider the conic C of equation $\{(x+y+z)^2 + xy = 0\}$ and the abelian covering ramified on the coordinate axis, then $\tilde{C}_n$ turns out to be $\{(x^n + y^n + z^n)^2 + x^n y^n = 0\}$, which can be deformed into $C_{2,n} \equiv \{(y^n - z^n)^2 + (x^2 - y^2)^n\}$.

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Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
851 S. Morgan Street
Chicago, Illinois, 60607-7045
USA
e-mail: jicogo@math.uic.edu

Departamento de Álgebra
Universidad Complutense de Madrid
Av. Ciudad Universitaria s/n
28040 Madrid
España
e-mail: jicogo@eucmos.sim.ucm.es